REDUCTION FORMULA Section - 6

6.1 Introduction

Some functions occur whose integrals are not immediately reducible to one or other of the standard form and whose integrals are not directly obtainable In such cases we find out a formula (called reduction formula) which connects one integral with another in which the intergrand is of the same type but is of lower degree or order and hence easier to integrate.

The following examples will illustrate you the concept of deriving the reduction formulas for various functions :

Illustration - 24 If $I_n = \int sin^n\, x\, dx$, then $I_n = \dfrac{-cos\, x\, sin^{n-1}\, x}{n} + f(n) I_{n-2}$, for $f(n) =$

(A) $\dfrac{n-1}{n}$ **(B)** $\dfrac{n}{n-1}$ **(C)** $\dfrac{n-2}{n}$ **(D)** $\dfrac{n}{n-2}$

SOLUTION : (A)

Let $I_n = \int \sin^n x\, dx = \int \sin^{n-1} x \cdot \sin x\, dx$

Apply by parts taking $\sin^{n-1} x$ as first part and $\sin x$ as second part.

$$\Rightarrow \quad I_n = \sin^{n-1} x \cdot (-\cos x) + \int (n-1) \sin^{n-2} x \cos^2 x\, dx$$

$$= -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x\, (1 - \sin^2 x)\, dx$$

$$= -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x\, dx - (n-1) \int \sin^n x\, dx$$

$$\Rightarrow \quad I_n = -\cos x \sin^{n-1} x + (n-1) I_{n-2} - (n-1) I_n$$

$$\Rightarrow \quad nI_n = -\cos x \sin^{n-1} x + (n-1) I_{n-2}; \qquad I_n = -\frac{\cos x \sin^{n-1} x}{n} + \frac{n-1}{n} I_{n-2}$$

Illustration - 25 If $I_n = \int \tan^n x\, dx$, then $(n-1)(I_n + I_{n-2}) =$

(A) $tan^{n-2}\, x$ **(B)** $tan^{n-1}\, x$ **(C)** $tan^n\, x$ **(D)** $tan^{n+1}\, x$

SOLUTION : (B)

Here $I_n = \int \tan^n x\, dx = \int \tan^{n-2} x \tan^2 x\, dx$

$$= \int \tan^{n-2} x\, (\sec^2 x - 1)\, dx$$

$$= \int \tan^{n-2} x \sec^2 x\, dx - I_{n-2}$$

$$\Rightarrow \quad I_n + I_{n-2} = \frac{\tan^{n-1} x}{n-1}$$

Hence $(n-1)(I_n + I_{n-2}) = \tan^{n-1} x$.

Illustration - 26 If $\int sec^n\, x\, dx = \dfrac{sec^{n-2}\, x\, tan\, x}{n-1} + f(n) \int sec^{n-2}\, x\, dx$, then $f(n) =$

(A) $\dfrac{n}{n-1}$ **(B)** $\dfrac{n-1}{n}$ **(C)** $\dfrac{n-2}{n-1}$ **(D)** $\dfrac{n-1}{n-2}$

SOLUTION : (C)

Let $\quad I_n = \int \sec^n x \quad \Rightarrow \quad I_n = \int \sec^{n-2} x \sec^2 x \, dx$

Apply by parts taking $\sec^{n-2} x$ as the first part and $\sec^2 x$ as the second part.

$$\Rightarrow \quad I_n = \sec^{n-2} x \int \sec^2 x \, dx - \int \left[\frac{d}{dx} (\sec^{n-2} x) \int \sec^2 x \, dx \right] dx$$

$$\Rightarrow \quad I_n = \sec^{n-2} x \tan x - \int (n-2) \sec^{n-3} x \sec x \tan x \tan x \, dx$$

$$\Rightarrow \quad I_n = \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) \, dx$$

$$\Rightarrow \quad I_n = \sec^{n-2} x \tan x - (n-2) \int \sec^n x \, dx + (n-2) \int \sec^{n-2} x \, dx$$

$$\Rightarrow \quad I_n + (n-2) I_n = \sec^{n-2} x \tan x + (n-2) \int \sec^{n-2} x \, dx.$$

$$\Rightarrow \quad (n-1) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$

Hence $\displaystyle \int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx.$

Illustration - 27 *If* $I_n = \int e^{ax} \cos^n x \, dx$, then $I_n = e^{ax} \left(\dfrac{f(x)}{a^2 + n^2} \right) \cos^{n-1} x + \dfrac{n(n-1)}{a^2 + n^2} I_{n-2}$ *for* $f(x) =$

(A) $a \cos x + n \sin x$ **(B)** $a \sin x + n \cos x$ **(C)** $a^2 \cos x + n^2 \sin x$ **(D)** $a^2 \sin x + n^2 \cos x$

SOLUTION : (A)

Let $\quad I_n = \int e^{ax} \cos^n x \, dx$

Apply by parts taking $\cos^n x$ as the first part and e^{ax} as the second part.

$$\Rightarrow \quad I_n = \cos^n x \int e^{ax} dx - \int \left[\frac{d}{dx} (\cos^n x) \int e^{ax} \, dx \right] dx$$

$$\Rightarrow \quad I_n = \frac{e^{ax}}{a} \cos^n x - \int n \cos^{n-1} x (-\sin x) \frac{e^{ax}}{a} \, dx$$

$$\Rightarrow \quad I_n = \frac{1}{a} e^{ax} \cos^n x + \frac{n}{a} \int (\cos^{n-1} x \sin x) \, e^{ax} dx$$

Apply by parts again taking $\cos^{n-1} x \sin x$ as first part and e^{ax} as second part.

$$\Rightarrow \quad I_n = \frac{1}{a} e^{ax} \cos^n x + \frac{n}{a} (\cos^{n-1} x \sin x) \int e^{ax} - \frac{n}{a} \int \left[\frac{d}{dx} (\cos^{n-1} x \sin x) \int e^{ax} dx \right] dx$$

$$\Rightarrow \quad I_n = \frac{1}{a} e^{ax} \cos^n x + \frac{n}{a} \cos^{n-1} x \sin x \frac{e^{ax}}{a} - \frac{n}{a} \int \left[-(n-1) \cos^{n-2} x \sin^2 x + \cos^{n-1} x \cos x \right] \frac{e^{ax}}{a} \, dx$$

$$\Rightarrow \quad I_n = \frac{1}{a} e^{ax} \cos^n x + \frac{n}{a^2} e^{ax} \cos^{n-1} x \sin x + \frac{n(n-1)}{a^2} \int e^{ax} \cos^{n-2} x (1 - \cos^2 x) \, dx - \frac{n}{a^2} \int e^{ax} \cos^n x \, dx$$

$$\Rightarrow \quad I_n = \frac{1}{a} e^{ax} \cos^n x + \frac{n}{a^2} e^{ax} \cos^{n-1} x \sin x + \frac{n(n-1)}{a^2} I_{n-2} - \frac{n^2}{a^2} I_n$$

$$\Rightarrow \quad \left(1 + \frac{n^2}{a^2} \right) I_n = \frac{1}{a^2} e^{ax} (a \cos x + n \sin x) \cos^{n-1} x + \frac{n(n-1)}{a^2} I_{n-2}$$

Hence $\int e^{ax} \cos^n x \, dx = e^{ax}\left(\frac{a \cos x + n \sin x}{a^2 + n^2}\right)\cos^{n-1} x + \frac{n(n-1)}{a^2 + n^2}\int e^{ax} \cos^{n-2} x \, dx$

This is the required reduction formula.

Illustration - 28 *If* $I_{m,\,n} = \int \cos^m x \sin nx \, dx = f(m, n) I_{m-1,\, n-1} - \frac{\cos^m x \cos nx}{m+n}$, *then* $f(m, n) =$

(A) $\frac{m}{m+n}$ **(B)** $\frac{n}{m+n}$ **(C)** $\frac{m-1}{m+n}$ **(D)** $\frac{n-1}{m+n}$

SOLUTION : (A)

Let $I_{m,\,n} = \int \cos^m x \sin nx \, dx$

Apply by parts taking $\cos^m x$ as the first part and $\sin nx$ as the second part.

$$\Rightarrow \quad I_{m,\,n} = \cos^m x\left(-\frac{\cos nx}{n}\right) - \int m \cos^{m-1} x(-\sin x)\left(-\frac{\cos nx}{n}\right) dx$$

$$\Rightarrow \quad I_{m,\,n} = -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^{m-1} x(\sin x \cos nx)\, dx$$

Now $\sin(n-1)x = \sin nx \cos x - \cos nx \sin x$

$\Rightarrow \quad \cos nx \sin x = \sin nx \cos x - \sin(n-1)x.$

$$\Rightarrow \quad I_{m,\,n} = -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^{m-1} x[\sin nx \cos x - \sin(n-1)x]\, dx$$

$$\Rightarrow \quad I_{m,\,n} = -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^m x \sin nx \, dx + \frac{m}{n}\int \cos^{m-1} x \sin(n-1)x \, dx$$

$$\Rightarrow \quad \left[1 + \frac{m}{n}\right] I_{m,\,n} = -\frac{\cos^m x \cos nx}{n} + \frac{m}{n} I_{m-1,\,n-1} \quad \Rightarrow \quad I_{m,\,n} = \frac{m}{m+n} I_{m-1,\,n-1} - \frac{\cos^m x \cos nx}{m+n}$$